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SECOND-ORDER DIFFERENTIAL GAME OF KIND*

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An algorithm is derived for solving a differential game of kind /1/ for a second-order conflict-controlled system. The article is closely related to /2-5/.

1. Let a conflict-controlled system's motion on the plane R^2 be described by the differential equation

$$y^{\cdot}(t) = Ay(t) + u(t) + v(t) \tag{1.1}$$

where A is a constant 2×2-matrix whose eigenvalues have a nonzero imaginary part, $u(t)$ is the first player's controlling parameter, $v(t)$ is that of the second player. At each instant t the parameter $u(t)$ is chosen from a segment $P \subset R^2$ and $v(t)$ is chosen from a convex compactum $Q \subset R^2$. The first player strives to take system (1.1) into a prescribed point m and the second player tries to prevent this. The symbol $\mathbf{U}$ denotes the set of strategies /2/ of the first player, namely, the set of all functions prescribed on $R_+ \times R^2$, with values in P. Here R_+ is the set of nonnegative numbers. The symbol $\mathbf{V}$ denotes the set of all measurable functions of time, with values in Q. Let Δ be an arbitrary partitioning of the semi-axis R_+ by points $0 = t_1 < t_2 < \cdots (t_i \to \infty$ as $i \to \infty)$ and $d(\Delta)$ be the partitioning's diameter. For fixed Δ, x, U, v by $y(\cdot; \Delta, x, U, v)$ we denote an absolutely continuous function of time prescribed on R_+ with values in R^2, equalling x when $t=0$ and being on each half-open interval $t_i \leqslant t < t_{i+1}$ $(i = 1, 2, \ldots)$ of partitioning Δ a solution of the differential equation

$$y^{\cdot}(t) = Ay(t) + U(t_i, y(t_i)) + v(t)$$

Let a_* and a^* be columns of matrix A, p_* and p^* be extreme points of segment P, and $s = (a_*, a^*, m, p_*, p^*, Q)$. By $B(s)$ we denote the collection of all $x \in R^2$ for each of which there exist a strategy $U \in \mathbf{U}$, an instant $\theta > 0$ and a mapping $\varepsilon \to \delta(\varepsilon)$ from R_+ into R_+, such that for any $\varepsilon > 0$, partitioning Δ with diameter $d(\Delta) \leqslant \delta(\varepsilon)$ and function $v \in \mathbf{V}$ we can find an instant $t \in [0, \theta]$ at which $y(t; \Delta, x, U, v)$ lies in the ε-neighborhood of point m. In other words, the set $B(s)$ is the collection of all initial points x on the plane, for each of which there exists a first player's feedback action method guaranteeing the transfer for system (1.1) from x to m in finite time under any actions by the second player.

If for a chosen s the function

$$\varphi(l) = \max_{p\in R}\min_{q\in Q} l'(p+q), \quad l \in R^2$$

is convex or concave, then the solving of the problem of seeking set $B(s)$ reduces /2/ to the solving of a corresponding control problem. Questions on the description of set $B(s)$ when the conditions of convexity or concavity of φ are not necessarily fulfilled where taken up in /3-5/. The present article relies on /4,5/. In it we derive, for the case when φ is not a convex or concave function, an algorithm for constructing a certain set $C(s)$ coinciding with $\mathrm{cl}\, B(s)$ if s belongs to the continuity set of the mapping $s \to \mathrm{cl}\, B(s)$ (cl is the symbol of closure in a Euclidean metric). In contrast to the one described in /5/ the algorithm proposed below admits of computer realization. On its basis V.L. Turova wrote a computer programme for the construction of set $C(s)$. Examples were run on a computer.

Let us make the concept of continuity of mapping $s \to \mathrm{cl}\, B(s)$ more precise. Let D be the collection of $(a_*, a^*) \in R^2 \times R^2$ such that the matrix $A = \| a_*, a^* \|$ has eigenvalues with nonzero imaginary part. The symbol X denotes the space of compact subsets of R^2, with the Hausdorff metric $\mathrm{dist}(\cdot,\cdot)$ /6/; the symbol Y denotes the set of all closed subsets of R^2. From the product $R^2 \times R^2 \times X$ we pick out the subset Π of elements (p_*, p^*, Q) for each of which the function φ is not convex or concave. Let $S = D \times R^2 \times \Pi$, $\mathrm{Dist}(\cdot,\cdot)$ be the Hausdorff metric in S. A mapping F from S into Y is said to be continuous at a point s if for any compactum $\Gamma \subset R^2$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathrm{dist}\ (F(s) \cap \Gamma, F(s_*) \cap \Gamma) \leqslant \varepsilon$ for every $s_* \in S$ satisfying the inequality $\mathrm{Dist}(s, s_*) \leqslant \delta$. Let $S_1 \subset S$ be the set of all points of continuity of the mapping $s \to \mathrm{cl}\, B(s)$ from S into Y. It can be shown that the set S_1 is open and that $S \diagdown S_1 \subset \mathrm{cl}_D S_1$ (cl_D is the symbol of closure in the metric $\mathrm{Dist}(\cdot,\cdot)$). Thus, the set $S \diagdown S_1$ is "small".

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2. We fix $s \in S_1$. Without loss of generality we take it that the phase trajectories of the equation $y^{\cdot}(t) = -Ay(t)$ go around the origin in the counterclockwise direction as t increases. We separate the plane into four convex cones K_i $(i = 1, 2, 3, 4)$, running in succession, with vertex at the origin, a nonempty interior and an opening $< \pi$ such that: 1) the restriction of φ to K_1 (K_3) is a concave function and the restriction to K_2 (K_4) is a convex function; 2) the restriction of φ to any cone K_i is not a linear function. The existence of such a separation follows from the definition of function φ and from the assumption (made in the definition of set S) that it is not convex or concave.

We fix and denote by the symbol E an arbitrary closed polygonal line on the plane, consisting of four links, such that if E_i is its link numbered i, then

$$\mathrm{cl}\, K_i = \bigcup_{\lambda \geqslant 0} \lambda E_i$$

Let H_0 be the restriction to $E \times R^2$ of the function

$$H(l, x) = l'Ax + \varphi(l), \; l \in R^2, \; x \in R^2$$

In terms of function H_0 we formulate the necessary and sufficient conditions for $B(s) \neq \{m\}$ /4/. For any l_1 and l_2 from E, by $\rho(l_1, l_2)$ we denote the angle between the vectors l_1 and l_2, taken counterclockwise from the first to the second. When $l_1 = l_2$ we set $\rho(l_1, l_2) = 0$. We write $l_1 < l_2$ if $l_1 \neq l_2$ and $\rho(l_1, l_2) < \pi$. We say that a vector $l^* \in E$ is a plus-to-minus zero of a real function f prescribed on E if $f(l^*) = 0$ and $f(l) > 0$ $(f(l) < 0)$ for any $l < l^*$ $(l^* < l)$ sufficiently close to l^*. In an analogous sense we speak of a minus-to-plus zero of function f. By the symbol F_1 we denote the collection of all $x \in R^2$ for each of which there exist l_* and l^* from E, being, respectively, the minus-to-plus and the plus-to-minus zeros of the function $H_0(\cdot, x)$, where $\rho(l_*, l^*) > \pi$ and $H_0(l, x) \neq 0$ for $l \in E$ different from l_* and l^*. We define the set F_2 as F_1 except that the condition $\rho(l_*, l^*) > \pi$ is replaced by $\rho(l_*, l^*) = \pi$. For all $l \in E$, $x \in R^2$, by $\Lambda(l, x)$ we denote the ray issuing from x, whose direction after a rotation by $\pi/2$ counterclockwise coincides with that of vector l. For $\varepsilon > 0$, $x \in R^2$ let $O(\varepsilon, x)$ denote the ε-neighborhood of point x.

Lemma. Let $s \in S$. Then the relation $B(s) \neq \{m\}$ is equivalent to one of the conditions: 1) $m \in F_1$, 2) $m \in F_2$ and $\varepsilon > 0$ exists such that $O(\varepsilon, m) \cap \Lambda(l^*, m) \subset F_1 \cup F_2$, where l^* is a plus-to-minus zero of function $H_0(\cdot, m)$.

If the lemma's condition 2) is fulfilled, then s is not a point of continuity of the mapping $s \to \mathrm{cl}\, B(s)$. Therefore, the computer verification of only the condition 1) is sensible. When it is fulfilled, we pass on to the construction of the curves defining the set $C(s)$. When condition 1) is not fulfilled, we set $C(s) = \{m\}$.

3. We introduce the concepts and notation needed. For any integer $1 \leqslant c \leqslant 5$ we set $(c) = c$ if $c \in \{1, 2, 3, 4\}$ and $(c) = 1$ if $c = 5$. Let $\bar{c} = 1$ when $c = 2$ and $\bar{c} = 2$ when $c = 1$. For $n = 1, 2$, $i = 1, 2$, $k = 1, 2$ we take

$$E_k^{(n)}(i) = E_{(2i+n+k-3)}, \qquad \Gamma^{(n)}(i) = E_1^{(n)}(i) \cup E_2^{(n)}(i)$$

$$P_k^{(n)}(i) = \{x \in R^2 \colon (-1)^{k+1} H_0^{(n)}(l, x) \leqslant 0, \; l \in E_k^{(n)}(i)\}$$

$$M^{(n)}(i) = R^2 \setminus (P_1^{(n)}(i) \cup P_2^{(n)}(i)), \qquad T_k^{(n)}(i) = \partial P_k^{(n)}(i) \setminus \partial P_{\bar{k}}^{(n)}(i)$$

Here ∂ is the symbol for boundary, $H_0^{(n)}(l, x) = (-1)^n H_0(l, x)$. By symbols $e_{k*}^{(n)}(i)$ and $e_k^{(n)*}(i)$ we denote the extreme points of segment $E_k^{(n)}(i)$. We take it that $e_{k*}^{(n)}(i) < e_k^{(n)*}(i)$. Let

$$\Gamma^{(n)}(i) = \Gamma^{(n)}(i) \setminus (\{e_{1*}^{(n)}(i)\} \cup \{e_2^{(n)*}(i)\})$$

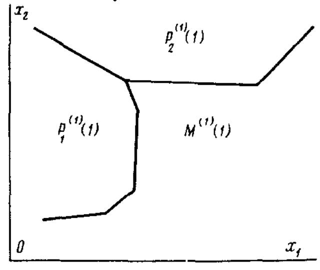

Fig.1

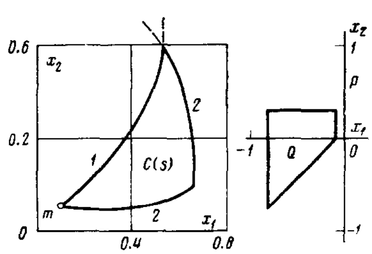

Fig.2

Fig.1 shows a possible form for the sets $P_1^{(1)}(1)$, $P_2^{(1)}(1)$, $M^{(1)}(1)$. For any $n = 1, 2$, $i = 1, 2$ a minus-to-plus zero of function $H_0^{(n)}(\cdot, x)$, belonging to $\Gamma^{(n)}(i)$ exists for every $x \in M^{(n)}(i)$. Function $H_0^{(n)}(\cdot, x)$ is convex on $E_1^{(n)}(i)$ and concave on $E_2^{(n)}(i)$; therefore, the vector from $\Gamma^{(n)}(i)$, being a minus-to-plus zero, is unique. We denote it by $L^{(n)}(i, x)$. If $x \overline{\in} M^{(n)}(i)$, then the function $H_0^{(n)}(\cdot, x)$ does not have the minus-to-plus zero in $\Gamma^{(n)}(i)$. The function $L^{(n)}(i, \cdot)$ satisfies a local Lipschitz condition in $M^{(n)}(i)$

For $n = 1, 2$, $l \in E$ let $\gamma^{(n)}(l)$ be the unit vector turned by $\pi/2$ relative to l, counterclockwise if $n = 1$ and clockwise if $n = 2$. We set $J^{(n)}(i, x) = \gamma^{(n)}(L^{(n)}(i, x))$. Let $n = 1, 2$, $i = 1, 2$, $x \in M^{(n)}(i) \cup T_1^{(n)}(i)$. By $q^{(n)}(\cdot, i, x)$ we denote a smooth function (a curve in parametric notation) defined on some segment $[0, \tau^{(n)}(i, x)]$, $\tau^{(n)}(i, x) > 0$, satisfying the conditions $q^{(n)}(0, i, x) = x$, $q^{(n)}(\tau^{(n)}(i, x), i, x) \in T_2^{(n)}(i)$ and being on $(0, \tau^{(n)}(i, x))$ a solution of the differential equation

$$\psi^{\cdot}(\tau) = J^{(n)}(i, \psi(\tau))$$

Such a function (curve) exists and is unique. The curve $q^{(n)}(\cdot, i, x)$ is a smooth semipermeable curve /1/. For the curve $q^{(n)}(\cdot, i, x)$ we introduce the concepts of sprout instant and point. By $W^{(n)}(i, x)$ we denote the collection of all $\tau \in (0, \tau^{(n)}(i, x))$ for each of which there exists $l^0 \in \Gamma^{(n)}(i) \setminus \{e_2^{(n)*}(i)\}$, such that

1) $\rho(L^{(n)}(i, q^{(n)}(\tau - 0, i, x)), l^0) < \pi$
2) $H_0^{(n)}(l, q^{(n)}(\tau, i, x)) \geqslant 0$, $l \in E$, $L^{(n)}(i, q^{(n)}(\tau - 0, i, x)) < l < l^0$
3) $H_0^{(n)}(l, q^{(n)}(\tau, i, x)) < 0$

for any $l \in E$ sufficiently close to l^0 and satisfying the relation $l^0 < l$. If set $W^{(n)}(i, x) \neq \phi$, then it consists of one or two elements. We denote the element closest to $\tau^{(n)}(i, x)$ by $w^{(n)}(i, x)$ and call it the sprout instant. The equality $w^{(n)}(i, x) = \tau^{(n)}(i, x)$ is possible. The point $r^{(n)}(i, x) = q^{(n)}(w^{(n)}(i, x), i, x)$ is called the sprout point.

4. Let the lemma's condition 1) be fulfilled. By $l_1^{(n)}$ we denote a minus-to-plus zero of function $H_0^{(n)}(\cdot, m)$, $n = 1, 2$. Let $i^{(n)} \in \{1, 2\}$ be such that $l_1^{(n)} \in \Gamma^{(n)}(i^{(n)}) \setminus \{e^{(n)*}(i^{(n)})\}$. If $i^{(2)} = i^{(1)}$, then a sprout point exists on curve $q^{(2)}(\cdot, i^{(2)}, m)$. We set

$$p_1^{(2)}(\tau) = \begin{cases} q^{(2)}(\tau, i^{(2)}, m), & \tau \in [0, w^{(2)}(i^{(2)}, m)) \\ q^{(2)}(\tau - w^{(2)}(i^{(2)}, m), \bar{i}^{(2)}, r^{(2)}(i^{(2)}, m)), & \tau \in [w^{(2)}(i^{(2)}, m) \\ \tau^{(2)}(\bar{i}^{(2)}, r^{(2)}(i^{(2)}, m))] & \end{cases}$$

If $i^{(2)} \neq i^{(1)}$, let $p_1^{(2)}(\tau) = q^{(2)}(\tau, i^{(2)}, m)$. By virtue of the special properties of point m the curve $p_1^{(2)}$ does not selfintersect. Having constructed $p_1^{(2)}$, we go on to construct the curve $p_1^{(1)}(\tau) = q^{(1)}(\tau, i^{(1)}, m)$. Let $[0, \Theta_1^{(1)}]$, $[0, \Theta_1^{(2)}]$ be the domains of curves $p_1^{(1)}$, $p_1^{(2)}$. Moving along $p_1^{(1)}$ from the point m, we verify the intersection of curve $p_1^{(1)}(0, \Theta_1^{(1)}]$ with the curve $p_1^{(2)}$. Here and later, for a function f of a real variable, the notation $f\,[a, b]$ ($f\,[a, b)$, $f\,(a, b]$) signifies the restriction of f to the segment $[a, b]$ (to the half-open intervals $[a, b)$, $(a, b]$). If there is an intersection, let $\alpha_1^{(1)} \in (0, \Theta_1^{(1)}]$ be the first instant of intersection and let $\alpha_1^{(2)} \in (0, \Theta_1^{(2)}]$ be such that $p_1^{(2)}(\alpha_1^{(2)}) = p_1^{(1)}(\alpha_1^{(1)})$.

We define $C(s)$ as a closed set bounded by a curve made up of the arcs $p_1^{(1)}[0, \alpha_1^{(1)}]$, $p_1^{(2)}[0, \alpha_1^{(2)}]$. When $p_1^{(1)}$ is not tangent to $p_1^{(2)}$ at the point of first intersection, then the boundary of set $C(s)$ is a piecewise-smooth semipermeable curve. In case of tangency the boundary is not a semipermeable curve and $C(s) \neq \mathrm{cl}\, B(s)$. In this case, however, the chosen $s \in S$ does not belong to the set S_1 of points of continuity of mapping $s \to \mathrm{cl}\, B(s)$. Fig.2 shows a possible form of set $C(s)$. The curves $p_1^{(1)}$, $p_1^{(2)}$ are labelled *1*, *2*. The arcs of the curves, not belonging to $\partial C(s)$, are shown by dashed lines. If the curve $p_1^{(1)}(0, \Theta_1^{(1)}]$ does not intersect curve $p_1^{(2)}$, the construction is continued. We find the sprout point on $p_1^{(2)}$, understanding by this the sprout point on curve $q^{(2)}(\cdot, \bar{i}^{(2)}, r^{(2)}(i^{(2)}, m))$ when $i^{(1)} = i^{(2)}$ and the sprout point on curve $q^{(2)}(\cdot, i^{(2)}, m)$ when $i^{(1)} \neq i^{(2)}$. When $p_1^{(1)}(0, \Theta_1^{(1)}]$ does not intersect $p_1^{(2)}$, such a point exists. Let $c_2^{(2)} = w^{(2)}(\bar{i}^{(2)}, r^{(2)}(i^{(2)}, m))$ when $i^{(1)} = i^{(2)}$ and $c_2^{(2)} = w^{(2)}(i^{(2)}, m)$ when $i^{(1)} \neq i^{(2)}$ and let $z_2^{(2)} = p_1^{(2)}(c_2^{(2)})$. We set

$$\theta_2^{(2)} = \tau^{(2)}(i^{(1)}, z_2^{(2)}), \qquad d_2^{(2)} = c_2^{(2)} + \theta_2^{(2)}$$

$$g_2^{(2)}(\tau) = \begin{cases} p_1^{(2)}(\tau), & \tau \in [0, c_2^{(2)}) \\ p_2^{(2)}(\tau - c_2^{(2)}), & \tau \in [c_2^{(2)}, d_2^{(2)}] \end{cases}$$

In case $i^{(1)} \neq i^{(2)}$ we proceed to section 1) and in case $i^{(1)} = i^{(2)}$, to section 2).

1) Moving along $p_2^{(2)}$ from $z_2^{(2)}$, we verify the intersection of $p_2^{(2)}$ with $p_1^{(1)}$. If there is one, let $k_2^{(2)} \in [0, \Theta_2^{(2)}]$ be the first instant of intersection and $\xi_2^{(2)} = k_2^{(2)} + c_2^{(2)}$, $\xi_2^{(1)} \in [0, \Theta_1^{(1)}]$

be such that $p_1^{(1)}(\xi_2^{(1)}) = p_2^{(2)}(k_2^{(2)})$. We define $C(s)$ as the closed set bounded by a curve composed of the arcs $p_1^{(1)}[0, \xi_2^{(1)}], g_2^{(2)}[0, \xi_2^{(2)}]$. If $p_2^{(2)}$ does not intersect $p_1^{(1)}$, then a sprout point exists on $p_1^{(1)}$. Let $c_2^{(1)} = w^{(1)}(i^{(1)}, m)$, $z_2^{(1)} = p_1^{(1)}(c_2^{(1)})$. We construct the curve $p_2^{(1)}(\tau) = q^{(1)}(\tau, \bar{i}^{(1)}, z_2^{(1)})$. We set

$$\Theta_2^{(1)} = \tau^{(1)}(\bar{i}^{(1)}, z_2^{(1)}), \quad d_2^{(1)} = c_2^{(1)} + \Theta_2^{(1)}$$

$$g_2^{(1)}(\tau) = \begin{cases} p_1^{(1)}(\tau), & \tau \in [0, c_2^{(1)}) \\ p_2^{(1)}(\tau - c_2^{(1)}), & \tau \in [c_2^{(1)}, d_2^{(1)}] \end{cases}$$

Moving along $p_2^{(1)}$ from $z_2^{(1)}$, we verify the intersection of $p_2^{(1)}$ with $g_2^{(2)}$. If there is one, let $a_2^{(1)} \in [0, \Theta_2^{(1)}]$ be the first instant of intersection and $\alpha_2^{(1)} = a_2^{(1)} + c_2^{(1)}$, $\alpha_2^{(2)} \in [0, d_2^{(2)}]$ be such that $g_2^{(2)}(\alpha_2^{(2)}) = p_2^{(1)}(a_2^{(1)})$. As $C(s)$ we take the closed set bounded by a curve composed of the arcs $g_2^{(1)}[0, \alpha_2^{(1)}]$, $g_2^{(2)}[0, \alpha_2^{(2)}]$. If $p_2^{(1)}$ does not intersect $g_2^{(2)}$, then a sprout point exists on $p_2^{(2)}$ We construct the curve $p_3^{(2)}(\tau) = q^{(2)}(\tau, \bar{i}^{(1)}, z_3^{(2)})$. We set

$$\Theta_3^{(2)} = \tau^{(2)}(\bar{i}^{(1)}, z_3^{(2)}), \quad d_3^{(2)} = c_2^{(2)} + c_3^{(2)} + \Theta_3^{(2)}$$

$$g_3^{(2)}(\tau) = \begin{cases} g_2^{(2)}(\tau), & \tau \in [0, c_2^{(2)} + c_3^{(2)}) \\ p_3^{(2)}(\tau - c_2^{(2)} - c_3^{(2)}), & \tau \in [c_2^{(2)} + c_3^{(2)}, d_3^{(2)}] \end{cases}$$

We go on to section 2).

2) Let $\omega = 2$ when $i^{(1)} = i^{(2)}$ and $\omega = 3$ when $i^{(1)} \neq i^{(2)}$. We say that an unwinding (of curve $g_\omega^{(2)}$) has been fixed on curve $p_\omega^{(2)}$ if $\tau^* \in [0, \Theta_\omega^{(2)}]$ exists such that

$$\frac{dp_\omega^{(2)}}{d\tau}(\tau^*) = \frac{dp_1^{(2)}}{d\tau}(0)$$

$$H\left(\frac{dp_\omega^{(2)}}{d\tau}(\tau^*), \quad p_\omega^{(2)}(\tau^*)\right) < H\left(\frac{dp_\omega^{(2)}}{d\tau}(\tau^*), m\right)$$

2a) Assume that an unwinding has not been fixed to $p_\omega^{(2)}$ The subsequent constructions are recurrently defined. Set $g_1^{(1)} = p_1^{(1)}$, $d_1^{(1)} = \Theta_1^{(1)}$, $c_0^{(n)} = c_1^{(n)} = 0$ $(n = 1, 2)$. Suppose that the curves $g_{r+1}^{(2)}$, $g_r^{(1)}$, $r + 1 \geqslant \omega$ have been constructed. Denote

$$\lambda_k^{(n)} = \sum_{0 \leqslant j \leqslant k} c_j^{(n)}, \quad n = 1, 2, \quad 0 \leqslant k \leqslant r$$

Moving along $p_{r+1}^{(2)}$ from $z_{r+1}^{(2)}$ we verify the intersection of $p_{r+1}^{(2)}$ with $g_r^{(1)}[\lambda_{r-1}^{(1)}, d_r^{(1)}]$. For the case

$$p_{r+1}^{(2)} \cap g_r^{(1)}[\lambda_{r-1}^{(1)}, \quad d_r^{(1)}] \neq \varnothing \tag{4.1}$$

i.e., when an intersection exists, let $k_{r+1}^{(2)} \in [0, \Theta_{r+1}^{(2)}]$ be the first instant of intersection,

$$\lambda_{r+1}^{(2)} = \lambda_r^{(2)} + c_{r+1}^{(2)}, \quad \xi_{r+1}^{(2)} = k_{r+1}^{(2)} + \lambda_{r+1}^{(2)}, \quad \xi_{r+1}^{(1)} \in [\lambda_{r-1}^{(1)}, \quad d_r^{(1)}]$$

be such that $g_r^{(1)}(\xi_{r+1}^{(1)}) = p_{r+1}^{(2)}(k_{r+1}^{(2)})$, G_{r+1}^{ξ} be a closed set bounded by a curve made up of the arcs $g_r^{(1)}[0, \xi_{r+1}^{(1)}]$, $g_{r+1}^{(2)}[0, \xi_{r+1}^{(2)}]$. The curve $p_r^{(1)}$ may not have a sprout point only when condition (4.1) is fulfilled. Then we assume

$$C(s) = G_{r+1}^{\xi} \tag{4.2}$$

If a sprout point does exist on $p_r^{(1)}$, we take $c_{r+1}^{(1)} = w^{(1)}(i_r, z_r^{(1)})$, $z_{r+1}^{(1)} = p_r^{(1)}(c_{r+1}^{(1)})$. Here and later $i_k = i^{(1)}$ for odd k and $i_k = \bar{i}^{(1)}$ for even k. We define set $C(s)$ by equality (4.2) if (4.1) holds and $c_{r+1}^{(1)} \leqslant \xi_{r+1}^{(1)}$. If condition (4.1) is not fulfilled, or is fulfilled but $c_{r+1}^{(1)} > \xi_{r+1}^{(1)}$, we go on to construct the curve $p_{r+1}^{(1)}(\tau) = q^{(1)}(\tau, i_{r+1}, z_{r+1}^{(1)})$. We set

$$\lambda_{r+1}^{(1)} = \lambda_r^{(1)} + c_{r+1}^{(1)}, \quad \Theta_{r+1}^{(1)} = \tau^{(1)}(i_{r+1}, z_{r+1}^{(1)}), \quad d_{r+1}^{(1)} = \lambda_{r+1}^{(1)} - \Theta_{r+1}^{(1)}$$

$$g_{r+1}^{(1)}(\tau) = \begin{cases} g_r^{(1)}(\tau), & \tau \in [0, \lambda_{r+1}^{(1)}) \\ p_{r+1}^{(1)}(\tau - \lambda_{r+1}^{(1)}), & \tau \in [\lambda_{r+1}^{(1)}, d_{r+1}^{(1)}] \end{cases}$$

Moving along $p_{r+1}^{(1)}$ from $z_{r+1}^{(1)}$ we verify the intersection of $p_{r+1}^{(1)}$ with $g_{r+1}^{(2)}[\lambda_r^{(2)}, d_{r+1}^{(2)}]$. For the case

$$p_{r+1}^{(1)} \cap g_{r+1}^{(2)}[\lambda_r^{(2)}, \quad d_{r+1}^{(2)}] \neq \varnothing \tag{4.3}$$

let $a_{r+1}^{(1)} \in [0, \Theta_{r+1}^{(1)}]$ be the first instant of intersection, $\alpha_{r+1}^{(1)} = a_{r+1}^{(1)} + \lambda_{r+1}^{(1)}$, $\alpha_{r+1}^{(2)} \in [\lambda_r^{(2)}, d_{r+1}^{(2)}]$ be such that $g_{r+1}^{(2)}(\alpha_{r+1}^{(2)}) = p_{r+1}^{(1)}(a_{r+1}^{(1)})$, G_{r+1}^{α} be a closed set bounded by a curve made up of the arcs $g_{r+1}^{(1)}[0, \alpha_{r+1}^{(1)}]$, $g_{r+1}^{(2)}[0, \alpha_{r+1}^{(2)}]$. Suppose that condition (4.3) is valid. When condition (4.1) is not fulfilled, or is fulfilled but $\alpha_{r+1}^{(2)} < \xi_{r+1}^{(2)}$, we set

$$C(s) = G_{r+1}^{\alpha} \tag{4.4}$$

Let condition (4.1) hold and let $\alpha_{r+1}^{(2)} \geqslant \xi_{r+1}^{(2)}$. If a sprout point exists on $p_{r+1}^{(1)}$ we assume $c_{r+2}^{(1)} = w^{(1)}(i_{r+1}, z_{r+1}^{(1)})$, $z_{r+2}^{(1)} = p_{r+1}^{(1)}(c_{r+2}^{(1)})$ and we construct the curve $p_{r+2}^{(1)}(\tau) = q^{(1)}(\tau, i_{r+2}, z_{r+2}^{(1)})$. If a sprout point exists on $p_{r+2}^{(1)}$, we assume $c_{r+3}^{(1)} = w^{(1)}(i_{r+2}, z_{r+2}^{(1)})$, $z_{r+3}^{(1)} = p_{r+2}^{(1)}(c_{r+3}^{(1)})$, we construct the curve $p_{r+3}^{(1)}(\tau) = q^{(1)}(\tau, i_{r+3}, z_{r+3}^{(1)})$ etc. Two possibilities exist. Either for some $h \geqslant r+1$ the curve $p_h^{(1)}$ does not have a sprout point or the process of successive construction of curves $p_{r+2}^{(1)}$, $p_{r+3}^{(1)}, \ldots$ is infinite. In the first case we define $C(s)$ by equality (4.2). In the second case the infinite curve

$$g_{\infty}^{(1)}(\tau) = p_k^{(1)}(\tau - \lambda_k^{(1)}), \quad \tau \in [\lambda_k^{(1)}, \lambda_{k+1}^{(1)}), \; k = 1, 2, \ldots \tag{4.5}$$
$$\lambda_k^{(1)} = \sum_{0 \leqslant j \leqslant k} c_j^{(1)}$$

is a twisting spiral winding down onto its own limit cycle. The open set bounded by the limit cycle is denoted $K^{(1)}$. We set $C(s) = G_{r+1}^{\xi} \setminus K^{(1)}$.

Suppose that condition (4.3) not be fulfilled If a sprout point exists on $p_{r+1}^{(2)}$ we take $c_{r+2}^{(2)} = w^{(2)}(\bar{i}_{r+1}, z_{r+1}^{(2)})$, $z_{r+2}^{(2)} = p_{r+1}^{(2)}(c_{r+2}^{(2)})$. Let condition (4.1) be valid and suppose that either there is no sprout point on $p_{r+1}^{(2)}$ or that there is one but $c_{r+2}^{(2)} \geqslant k_{r+2}^{(2)}$. Then the subsequent constructions are carried out as described above when (4.1) and (4.3) are fulfilled and when $\alpha_{r+2}^{(2)} \geqslant \xi_{r+1}^{(2)}$, with the difference that when constructing the curve $p_{r+2}^{(1)}$ we verify its intersection with the curve $g_{r+1}^{(2)}[\lambda_{r+1}^{(2)}, d_{r+1}^{(2)}] = p_{r+1}^{(2)}$ and, if there is one, we bound the set $C(s)$ by a curve composed of the arcs $g_{r+2}^{(1)}[0, \alpha_{r+2}^{(1)}]$, $g_{r+1}^{(2)}[0, \alpha_{r+2}^{(2)}]$. Here $\alpha_{r+2}^{(1)} = a_{r+2}^{(1)} + \lambda_{r+2}^{(1)}$, $a_{r+2}^{(1)} \in [0, \Theta_{r+2}^{(1)}]$ is the first instant of intersection and $\alpha_{r+2}^{(2)} \in [\lambda_{r+1}^{(2)}, d_{r+1}^{(2)}]$ is such that $g_{r+1}^{(2)}(\alpha_{r+2}^{(2)}) = p_{r+2}^{(1)}(a_{r+2}^{(1)})$. When condition (4.1) is not fulfilled, a sprout point exists on $p_{r+1}^{(2)}$. Suppose that condition (4.1) is not fulfilled or is fulfilled but a sprout point exists on $p_{r+1}^{(2)}$ and that $c_{r+2}^{(2)} < k_{r+2}^{(2)}$. We construct the curve $p_{r+2}^{(2)}(\tau) = q^{(2)}(\tau, \bar{i}_{r+2}, z_{r+2}^{(2)})$. We assume

$$\Theta_{r+2}^{(2)} = \tau^{(2)}(\bar{i}_{r+2}, z_{r+2}^{(2)}), \quad \lambda_{r+2}^{(2)} = \lambda_{r+1}^{(2)} + c_{r+2}^{(2)}, \quad d_{r+2}^{(2)} = \lambda_{r+2}^{(2)} + \Theta_{r+2}^{(2)}$$

$$g_{r+2}^{(2)}(\tau) = \begin{cases} g_{r+1}^{(2)}(\tau), & \tau \in [0, \lambda_{r+2}^{(2)}) \\ p_{r+2}^{(2)}(\tau - \lambda_{r+2}^{(2)}), & \tau \in [\lambda_{r+2}^{(2)}, d_{r+2}^{(2)}] \end{cases}$$

Thus, in the last case we have obtained a transition form the curves $g_{r+1}^{(2)}, g_r^{(1)}$ to the curves $g_{r+2}^{(2)}, g_{r+1}^{(1)}$. With recurrent construction there can be only a finite number of such transitions.

2b) Suppose that an unwinding has been fixed on $p_{\omega}^{(2)}$. In this case the curve $p_{\omega}^{(2)}$ has a sprout point; we denote it $z_{\omega+1}^{(2)}$. The curve $p_{\omega+1}^{(2)}(\tau) = q^{(2)}(\tau, \bar{i}_{\omega+1}, z_{\omega+1}^{(2)})$ too has a sprout point, etc. The infinite curve

$$g_{\infty}^{(2)}(\tau) = p_k^{(2)}(\tau - \lambda_k^{(2)}), \quad \tau \in [\lambda_k^{(2)}, \lambda_{k+1}^{(2)}), \quad k = 1, 2, \ldots$$

(the notation is clear from the preceding exposition) is an untwisting spiral. If it has a limit cycle, let $K^{(2)}$ be the closed set bounded by it. When there is not limit cycle, we take $K^{(2)} = R^2$. If a sprout point does not exist on $p_{\omega-1}^{(1)}$ we set $C(s) = K^{(2)}$. Let a sprout point exist on $p_{\omega-1}^{(1)}$. We construct the curve $p_{\omega}^{(1)}$. When $\omega = 3$ we proceed to section 3). When $\omega = 2$ we verify the intersection of $p_2^{(1)}$ with $g_2^{(2)}(0, d_2^{(2)}]$ When it exists, we define $C(s)$ by equality (4.4), having set $r+1=2$ in it. Let there be no intersection. If a sprout point does not exist on $p_2^{(1)}$, we set $C(s) = K^{(2)}$. If a sprout point does exist, we construct the curve $p_3^{(1)}$ and proceed to section 3).

3) We say that an unwinding (of curve $g_3^{(1)}$) has been fixed on the curve $p_3^{(1)}$ if $\tau^* \in [0, \Theta_3^{(1)}]$, exists such that

$$\frac{dp_3^{(1)}}{d\tau}(\tau^*) = \frac{dp_1^{(1)}}{d\tau}(0), \quad H\left(\frac{dp_3^{(1)}}{d\tau}(\tau^*), p_3^{(1)}(\tau^*)\right) \leqslant H\left(\frac{dp_3^{(1)}}{d\tau}(\tau^*), m\right)$$

3a) Suppose that an unwinding has not been fixed on $p_3^{(1)}$ We verify the intersection of $p_3^{(1)}$ with $g_3^{(2)}[\lambda_2^{(2)}, d_3^{(2)}]$. When it exists we define $C(s)$ by the equality (4.4), having set $r+1=3$ in it. Let there be no intersection. We proceed to the construction of curve $p_4^{(1)}$, next $p_5^{(1)}$, etc. We take $C(s) = K^{(2)}$ if the process of successive construction of the curves $p_3^{(1)}, p_4^{(1)}, p_5^{(1)}, \ldots$ stops at a finite number. If it is infinite, the curve $g_{\infty}^{(1)}$ introduced by formula (4.5) is a twisting spiral winding down onto its own limit cycle. We set $C(s) = K^{(2)} \setminus K^{(1)}$ ($K^{(1)}$ is an open set bounded by the limit cycle of curve $g_{\infty}^{(1)}$).

3b) Suppose that an unwinding has been fixed on $p_3^{(1)}$. Then the curve $p_3^{(1)}$ has a sprout point, the curve $p_4^{(1)}$ too has a sprout point, etc. We define a recurrent method for constructing the curves. Suppose that the curves $g_{r+1}^{(2)}$, $g_{r+1}^{(1)}$, $r+1 \geqslant 3$ have been constructed. Moving along $p_{r+1}^{(1)}$ from $z_{r+1}^{(1)}$, we verify the intersection $p_{r+1}^{(1)}$ with $g_{r+1}^{(2)}[\lambda_r^{(2)}, d_{r+1}^{(2)}]$ (α-intersection) and the intersection of $p_{r+1}^{(1)}$ with $g_{r+1}^{(2)}[\lambda_{r-2}^{(2)}, \lambda_r^{(2)})$ (β-intersection). If the β-intersection exists, let $b_{r+1}^{(1)} \in [0, \Theta_{r+1}^{(1)}]$ be the first instant of β-intersection, $\beta_{r+1}^{(1)} = b_{r+1}^{(1)} + \lambda_{r+1}^{(1)}$, $\beta_{r+1}^{(2)} \in [\lambda_{r-2}^{(2)}, \lambda_r^{(2)})$ be such that $g_{r+1}^{(2)}(\beta_{r+1}^{(2)}) = p_{r+1}^{(1)}(b_{r+1}^{(1)})$, G_{r+1}^{β} be an open set bounded by a curve composed of the arcs $g_{r+1}^{(1)}[0, \beta_{r+1}^{(1)}]$, $g_{r+1}^{(2)}[0, \beta_{r+1}^{(2)}]$.

Suppose that α-intersection exists, but β-intersection does not exist, or both types of intersection exist but $\alpha_{r+1}^{(1)} \leqslant \beta_{r+1}^{(1)}$. Then we set $C(s) = G_{r+1}^{\alpha}$ (the instant $\alpha_{r+1}^{(1)}$ and the set G_{r+1}^{α} were introduced in the text below formula (4.3)). Suppose that β-intersection exists, but α-intersection does not exist, or both types of intersection exist but $\alpha_{r+1}^{(1)} > \beta_{r+1}^{(1)}$. We set $C(s) = K^{(2)} \setminus G_{r+1}^{\beta}$. If there is neither α- nor β-intersection, then we take the curves $g_{r+2}^{(2)}$ and $g_{r+2}^{(1)}$ as constructed. If for any $k \geqslant 3$ the curve $p_k^{(1)}$ has neither an α- nor a β-intersection, we define $C(s)$ as a closed set contained between the curves $g_\infty^{(1)}$, $g_\infty^{(2)}$. In this case the curves $g_\infty^{(1)}$, $g_\infty^{(2)}$ do not have limit cycles.

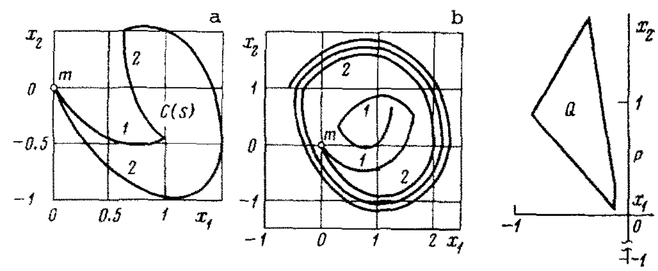

Fig.3

Figs.2 and 3 show the results of the calculation of three examples on a computer. In the first example (Fig.2) $m = (0.1; -0.1)$, in the second (Fig.3,a) and in the third (Fig.3,b) $m = (0; 0)$. In all examples P is a segment of length 2 on the x_2-axis, symmetric relative to the origin. The vertices of polygon Q: $\{(-0.84; -0.80), (-0.08; 0.31), (-0.08; 0.00)\}$ in the first example, $\{(-0.84; 0.90), (-0.36; 1.75), (-0.10; 0.25), (-0.10; 0.06)\}$ in the second and third. Matrix A has the form

$$\begin{Vmatrix} 0.1 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0.465 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0.05 & 1 \\ -1 & 0 \end{Vmatrix}$$

In the first example the set $C(s)$ is bounded by the curves $p_1^{(1)}[0, \alpha_1^{(1)}]$, $p_1^{(2)}[0, \alpha_1^{(2)}]$ (labelled 1, 2 on Fig.2), in the second, by the curves $p_1^{(1)}[0, \xi_2^{(1)}]$, $g_2^{(2)}[0, \xi_2^{(2)}]$ (labelled 1, 2) on Fig.3,a). In the third example $C(s)$ is bounded by the limit cycle of curve $g_\infty^{(2)}$, labelled 2 on Fig.3,b; the curve $g_3^{(1)}$ is labelled 1. The curve $p_3^{(1)}$ in the third example does not have a sprout.

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